Bayesian Inverse Problem for Linear System

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Abstract

This project investigates the inverse problem for linear systems as it poses an important aspect in real life applications such as medical imaging, astronomy, and geophysics. We attempt to find the unknown coefficients of a linear system, given a noisy observation on the solution of the linear system. The unknown coefficients are assumed to be dependent on a parameter which is in a prior probability space. The noise follows a probability distribution. We aim to find the posterior probability of the unknown given a set of observation using the Bayesian framework.

Definition (Bayesian Framework)

Let U be a parameter space and $\mathbb{R}^n$ be the data space. The relationship of U and $\mathbb{R}^n$ is defined as

$$G : U \longrightarrow \mathbb{R}^n$$

where G is the observation operator. In the practical settings, the observational data usually suffers from noise, η. The Bayesian framework express the problem with the following model

$$\delta = G(u) + \eta \tag{1}$$

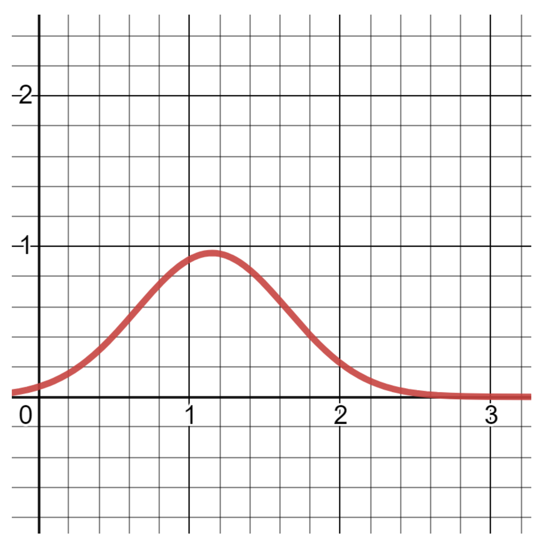
With equation (1), we aim to solve u given the observational data, δ. Hence, Bayesian approach is adopted by assuming that u and η follows some probability space respectively. The problem is then reduced to finding the posterior probability of u given δ.

$$\mathbb{P}(u|\delta) \propto \mathbb{P}(\delta|u)\mathbb{P}(u)$$

If $G(u)$ in (1) is continuous, then the posterior measure and the prior measure is related according to the Radon-Nikodym derivative.

$$\frac{d\rho^{\delta}}{d\rho}(u) \propto \exp(-\Phi(u;\delta))$$

where $\Phi(u;\delta)$ is the Bayesian Potential.



Linear System

Defined the Linear System, we can convert the Linear Inverse Problem with respect to the Bayes Rule

$$\vec{x} = A^{-1}(\vec{u})\vec{b} + \eta \tag{2}$$

Central Theorem

Bayes theorem is defined as follows

$$\mathbb{P}(u|\delta) \propto \mathbb{P}(\delta|u)\mathbb{P}(u)$$

where

1. $\mathbb{P}(u|\delta)$ is the posterior measure of u given δ
2. $\mathbb{P}(\delta|u)$ is the likelihood function of δ given u
3. $\mathbb{P}(u)$ is the prior measure with respect to u

Linear System

Let A be a $n \times n$ real matrix. Let $\vec{x}$ be the solution set of the set of variables of the linear equations and $\vec{b}$ be a given data. We consider the Linear System as such

$$A\vec{x} = \vec{b}$$

We assume that A is a symmetric positive definite matrix and is dependent on a parameter u that follows a prior probability space. Observation on the solution $\vec{x}$, contains noise, η which follows a Gaussian distribution, $\eta \sim \mathcal{N}(0, \Sigma)$ where Σ is a symmetric positive definite matrix.

Linear Inverse Problem

Assume that A can be represented as

$$A(\vec{u}) = A_0(\vec{u}) + \sum_{i\in\mathbb{N}} A_i(\vec{u})$$

for $\vec{u} \in U = \times_{i\in\mathbb{N}}[-1,1]$. Linear System: Given $\vec{b}$ and $\vec{x}+\eta$, find the posterior probability of the unknown parameter u in terms of the Radon-Nikodym derivative

$$\frac{d\rho^{\vec{x}}}{d\rho} \propto \exp\left(-\frac{1}{2}\left\|\vec{x} - A^{-1}(\vec{u})\vec{b}\right\|_{\Sigma}^{2}\right) \tag{3}$$

Well-posedness of Linear Inverse Problem
Define the Hellinger metric

$$\mathcal{H}^2(\rho^{\vec{x}}, \rho^{\vec{x}'}) = \frac{1}{2}\int_U \left(\sqrt{\frac{d\rho^{\vec{x}}}{d\rho}} - \sqrt{\frac{d\rho^{\vec{x}'}}{d\rho}}\right)^2 d\rho(u)$$

Along with (3), we have

$$\mathcal{H}^2(\rho^{\vec{x}}, \rho^{\vec{x}'}) \propto \frac{1}{2}\int_U \left[exp(-\frac{1}{2}\left\|\vec{x} - A^{-1}(u)\vec{b}\right\|_{\Sigma}^{2}) - \exp\left(-\frac{1}{2}\left\|\vec{x'} - A^{-1}(u)\vec{b}\right\|_{\Sigma}^{2}\right)\right]^2 d\rho(u)$$

With some algebraic manipulations, and the idea of diagonalisation of A, we get

$$\mathcal{H}(\rho^{\vec{x}}, \rho^{\vec{x}'}) \leqslant C\left\|\vec{x} - \vec{x'}\right\|_{\Sigma}$$

Error Bounding
Truncating the sums so that it can be used to approximate it into the finite-dimensional settings for easy computation. We have $A^J = A_0 + \sum_{i=1}^{J} u_i A_i$. We will show the error bound, with Hellinger distance between the approximated posterior and true posterior measures for a fixed vector $\vec{x}$. Working with the assumption that $\left\|A(\vec{u}) - A^J(\vec{u})\right\|_{\infty} \leqslant CJ^{-q}$ for some $q > 0$, we can show that

$$\mathcal{H}(\rho^{\vec{x}}, \rho^{\vec{x},J}) \leqslant CJ^{-q}$$

The Approach

All this under the uniform distribution, $U = \times_{i\in\mathbb{N}}[-1,1]$

1. Show that the inverse problem is well-posed by using continuity properties of the posterior measure of u via the Hellinger metric for $\vec{x}$ and $\vec{x'}$
2. Using partial sums of A^J for approximation of A, derive a satisfying error bound using the Hellinger distance between the truncated posterior measure and true posterior measure.

References

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